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Synchronization of musical words

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Abstract

We study the synchronization of musical sequences by means of an operation defined on finite or infinite words called superimposition. This operation can formalize basic musical structures such as melodic canons and serial counterpoint. In the case of circular canons, we introduce the superimposition of infinite words, and we present an enumeration algorithm involving Lyndon words, which appear to be a useful tool for enumerating periodic musical structures. We also define the superimposition of finite words, the superimposition of languages, and the iterated superimposition of a language, which is applied to the study of basic aspects of serial music. This leads to the study of closure properties of rational languages of finite words under superimposition and iterated superimposition. The rationality of the transduction associated with the superimposition appears to be a powerful argument in the proof of these properties. Since the superimposition of finite words is the max operation of a sup-semilattice, the last section addresses the link between the rationality of a sup-semilattice operation and the rationality of the order relation associated with it.

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1. Introduction

There exist many different ways of representing musical sequences as words over a finite alphabet, and such representations have been used extensively since the early days of computer music. The first models dealing with music sequences represented as words were probably Markov chains and probabilistic automata, used as far back as the end of the fifties and the beginning of the sixties, in automatic composition programs developed by Lejaren Hiller, Xenakis and Barbaud (see [14]).

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In the seventies, studies in automatic music generation or analysis focused on formal grammars, and many types of musical grammars have been proposed (see [26] for a survey). More recently, the generative power of musical grammars continued to be investigated, as in Steedman's 1984 paper on generative grammar for jazz chord sequences, which is a classical contribution to this field [28] (see [14] for further references on this subject).

There are probably many reasons why automata and grammars are useful models for music formalization, but one of them may be the fact that repetition is one of the most important features of music. Nicolas Ruwet, a musicologist influenced by linguistics, proposed in the sixties a method for music analysis based on criteria of repetition [27]. This idea has grown up in several directions, in computer applications involving formal models such as the “descriptive complexity” proposed in [6]. Recently a collection of approximate distributed matching problems arises in applications related to music indexation on the Internet [17].

In the field of automatic music generation, a new paradigm introduced by Gérard Assayag and Shlomo Dubnov in [3] consists in applying methods derived from the Lempel-Ziv compression algorithm in order to produce music. The idea is to build a dictionary of motives occurring in a given musical sequence, and to transform this dictionary into a successor function which chooses a new musical element following a current past context. This model makes it possible to generate new musical sequences that are amazingly close to the style of the original one, and it has been implemented in various systems such as [24].

In this paper, we extend the representation of musical sequences as words over a finite alphabet, by taking into account the fact that events in a musical sequence may occur *simultaneously*. As shown in [8,15], this can be achieved by considering an alphabet not reduced to single letters, but containing letters that associate simultaneous events. More precisely, we will assume that the alphabet is either the Cartesian product of k sets of events (in the case of infinite k -voice polyphonies), or the power set of a given set of events (in the case of finite words). Thus words are sequences of letters, which contain events. Events occurring in the same letter are considered as simultaneous events, and events occurring in successive letters are considered as successive events. In this way, it is possible to linearize any multivoice music data, and transform it into a linear stream of musical objects.

We study the synchronization of musical sequences by means of an operation on words called *superimposition*. This operation can easily formalize basic musical structures such as circular melodic canons or serial sequences. We provide constructions of these structures based on formal properties of the superimposition. After basic definitions (Section 2), the paper is divided into two parts dealing respectively with infinite and finite words.

The first part (Sections 3 and 4) defines the superimposition of infinite words, in order to analyze *circular melodic canons*. This part of the paper is motivated by ethnomusicological researches made on the harp music of Nzakara people from Central African Republic, where one can find two-voice melodic canons. Section 4 presents an algorithm based on finite difference calculus, which computes a particular type of periodic canons found in the Nzakara repertoire. Since two canons being

cyclic shifts of one another are not distinguished, this algorithm makes use of Lyndon words.

The second part (Sections 5–7) introduces the superimposition of finite words, which is extended to the superimposition of two languages and the iterated superimposition of a language. The latter one is applied to the formalization of the *basic rule of serial music*. Since the rational languages are closed by iterated superimposition (as established in Section 6), it follows that the basic serial language is rational.

The last two sections are devoted to closure properties of rational languages under superimposition and iterated superimposition. The superimposition of two rational languages is rational, and we compare this result to a similar one concerning the literal shuffle. These results involve the transductions associated with these operations. The fact that the transduction is rational appears to be a powerful argument in the proof of closure properties. Another important result asserts that the iterated superimposition of a rational language is rational, a result which is generalized to an arbitrary max operation of a semi-lattice defining a rational transduction. Section 7 investigates the links between the rationality of the max operation of a sup-semilattice, and the rationality of the order relation which is canonically associated with it.

2. Preliminaries

2.1. Finite and infinite words

Finite words over an alphabet A are defined as finite sequences of symbols from A . For a finite word w , we denote by $|w|$ the length of w . The concatenation of finite words is an associative operation defined on the set of finite words A^* . A subset of A^* is called a *language*. The empty word denoted by 1 is a neutral element for concatenation, so that A^* is a monoid (the free monoid generated by A). Infinite words over A are defined as functions from $\mathbb{N}$ to A .

Our interest for infinite words in this paper is focused on the notion of periodicity. For an infinite word u , we define the translated word Tu by

$$Tu(n) = u(n + 1)$$

for any integer n . We say that u is m -periodic if $u(n + m) = u(n)$ for any integer n , which is equivalent to the relation $T^m u = u$. If u is m -periodic, then Tu is also m -periodic. We define the *period* of u as the least integer m such that u is m -periodic.

For two periodic infinite words, we say that they are *conjugate* if one is the prefix of the other. This can be written

$$v = T^k u$$

for an integer k . The prefix relation is reflexive and transitive. For periodic infinite words, it is also symmetric, since $v = T^k u$ and $T^m u = u$ implies $T^{(m-1)k} v = T^{(m-1)k} T^k u = T^{mk} u = u$. Thus the conjugacy relation on infinite periodic words is an equivalence relation.

For finite words, we say that two words are *conjugate* if they are cyclic shifts of one another. It is easy to verify that two infinite m -periodic words are conjugate if and only if their finite prefixes of length m are conjugate.

A *cross-section* regarding an equivalence relation for a set X , is a subset S_X of X such that every element of X is equivalent to an element of S_X , and S_X does not contain two equivalent elements. Thus cross-sections regarding the conjugacy relation of infinite m -periodic words are isomorphic to cross-sections regarding the conjugacy relation of their finite prefixes of length m .

A *Lyndon word* is a finite primitive word which is minimal for the lexicographic order in its conjugacy class. Recall that a word is primitive when it is not a power of another word (for instance $ababab = (ab)^3$ is not primitive). The minimal element being unique, Lyndon words provide a tool for computing a cross-section for the conjugacy relation on finite words. The notion of Lyndon words appears to be important in music formalization, since loop structures are a fundamental feature of music. As we shall see in this paper, the enumeration of cyclic musical structures, such as those one can find in African music, relies deeply on Lyndon words.

Lyndon words can be computed efficiently. Jean-Pierre Duval has given an algorithm which gives the list in order of all Lyndon words of a given length, and his algorithm is optimal in the sense that computing each new Lyndon word in the list is done in linear time and with no auxiliary memory [18].

2.2. Finite automata

As recalled in the introduction, automata are suitable models for music formalization. An *automaton* is a graph with two subsets I and T of particular vertices called initial and final states respectively, and transitions between vertices labeled by letters from the alphabet A . A path in the automaton is a sequence of consecutive transitions. A finite word is *computed* (or *recognized*) by an automaton if it is the label of a path beginning at an element of I and ending at an element of T . We denote by $q^{-1}T$ the set of labels of paths going from q to an element of T .

The *de Bruijn graph* of order n is a particular automaton used in the computation of infinite words. Its set of vertices is the set of finite words of length n . For any letters a and b , and any word s of length $n - 1$, there is a transition from bs to sa labeled by a . The notion of path is extended to infinite paths, that is infinite sequences of consecutive transitions. The *computation* of an infinite word w by the de Bruijn graph of order n is the infinite path in the graph having w as its label. Note that this path contains every factor of w of length n .

We shall prove in this paper closure properties under superimposition for languages of finite words computed by an automaton. This requires some general definitions and results borrowed from [5] and [19] concerning the family of rational and recognizable subsets of an arbitrary monoid M , denoted respectively by $\text{Rat}(M)$ and $\text{Rec}(M)$.

The family $\text{Rat}(M)$ of *rational* subsets of M is the least family of subsets of M containing the empty set and the finite subsets, and closed under union, product and star operations. The direct image of a rational subset by a morphism is rational. A *rational*

expression is obtained by applying recursively union and star operations (respectively denoted by $+$ and $*$) to finite subsets of M .

The family $\text{Rec}(M)$ of *recognizable* subsets of M is the family of subsets X of M such that there exists only a finite number of sets $m^{-1}X = \{z, mz \in X\}$ for all m in M . The inverse image of a recognizable subset by a morphism is recognizable. The family $\text{Rec}(M)$ is closed under the Boolean operations union, intersection and complementary.

In the case of the free monoid A^* , Kleene's theorem asserts that one has $\text{Rat}(A^*) = \text{Rec}(A^*)$, and that every language of this family, which is called *regular*, can be computed by an automaton.

2.3. Transductions

We shall make use of the notion of transduction, as a technical tool simplifying the proof of closure properties. A *transduction* t from monoid M to monoid M' is a subset of $M \times M'$. The transduction is said to be *rational* if t belongs to $\text{Rat}(M \times M')$. Rational transductions are characterized by the decomposition theorem due to Nivat [23], asserting that t belongs to $\text{Rat}(M \times M')$ if and only if there exist an alphabet Z , a regular language K of Z^* , and morphisms φ and ψ from Z^* to M and M' such that $t = (\varphi, \psi)(K)$.

The two following properties will be used later, the first one being a direct consequence of Nivat's theorem since the image of a language X by a rational transduction t may be written as $t(X) = \psi(\varphi^{-1}(X) \cap K)$.

- (i) For any rational transduction t in $\text{Rat}(M \times M')$, if X is an element of $\text{Rec}(M)$, then $t(X)$ is an element of $\text{Rat}(M')$.
- (ii) If X and Y are regular languages of A^* , then $X \times Y$ is an element of $\text{Rec}(A^* \times A^*)$ (Mezei, [5, p. 54]).

Part I: Synchronizing infinite words

3. Circular melodic canons

We define the superimposition of infinite words. Let $E_0, \dots, E_{k-1}$ be sets of events, and B a particular subset of the Cartesian product $E_0 \times \dots \times E_{k-1}$. We shall say that an infinite word w over the alphabet B is the *superimposition* of k infinite words $v_0, v_1, \dots, v_{k-1}$ over the alphabets $E_0, \dots, E_{k-1}$, which is denoted by

$$w = \parallel_{0 \leq j \leq k-1} v_j$$

if and only if

$$w(n) = (v_0(n), v_1(n), \dots, v_{k-1}(n))$$

for any integer n . In this case, w is called a *k-voice polyphony*.

This definition is applied to the study of circular melodic canons, which are defined as infinite periodic words, and we compute a solution to the problem of building

such canons. This problem consists of satisfying simultaneously two constraints. First the melody must repeat itself but each repetition must have as long a length m as possible (the period of the circular canon). Second each note at a given time is the combination of the leading voice and the remaining voices at distance a multiple of p and determined unambiguously from the leading voice. As we shall see, the problem may be posed in terms of paths in a finite graph, and we will show that in the case of certain African canons, there is no way to satisfy these two conditions without violating at least once the rules. First, we begin with the general case of infinite canons which are not supposed to be periodic.

Let E_{k-1} be the set of events of the leading voice of the canon. For any k -tuple x we denote by x^+ its component in E_{k-1} . We assume that k mappings $t_0, \dots, t_{k-1}$ are given from E_{k-1} to the sets of events $E_0, \dots, E_{k-1}$ respectively, with t_{k-1} being the identity $t_{k-1} = \text{Id}$. These mappings are extended in a natural way to infinite words over the alphabet E_{k-1} .

A k -voice melodic canon with distance p is an infinite word u over the alphabet B such that there exists an infinite word v over E_{k-1} satisfying

$$u = \bigsqcup_{0 \leq j \leq k-1} T^{jp} t_j(v). \quad (3.1)$$

This means that each voice of the polyphony is deduced from voice v (the leading voice of the canon) through a musical transposition of its elements by the mapping t_j . Moreover, each voice is synchronized with v with a delay of $((k-1)-j) \cdot p$ time units.

Equivalently, one can say that the k -tuple $u(n)$ of the polyphony depends on k events occurring in its leading voice v , as expressed by the following equality

$$u(n) = (t_0(v(n)), t_1(v(n+p)), \dots, t_{k-2}(v(n+(k-2)p)), v(n+(k-1)p)),$$

which holds for any integer n .

For $k > 2$, a more general definition would consist in replacing integers $p, \dots, (k-2)p, (k-1)p$ by arbitrary integers $p_1, \dots, p_{k-2}, p_{k-1}$. The definition above corresponds to the case of “equidistant” voices. The theory of melodic canons presented in this section is restricted to this case, which includes most of the musical applications.

Example. This part of the paper is motivated by ethnomusicological researches made on harp music of Nzakara people from Central African Republic, where one can find such melodic canons, but our theory applies to Western classical music as well. In the case of Nzakara harp canons, the value of k is 2 (two-voice canons). Fig. 1 shows one of these two-voice melodic “canons”. The upper voice is reproduced, with just a few exceptions, in the lower voice, with a retardation as indicated by the two broken lines. These canon formulas have been studied in detail in [10,13], and one can hear some of them in [11,12].

The set of events E is equal to the set of the five strings of the harp denoted in ascending order $E = \{c_1, c_2, c_3, c_4, c_5\}$. The alphabet is a subset B of the Cartesian product $E \times E$, restricted to the five combinations of strings plucked simultaneously, actually played by Nzakara musicians, as shown in Fig. 1, denoted as integers

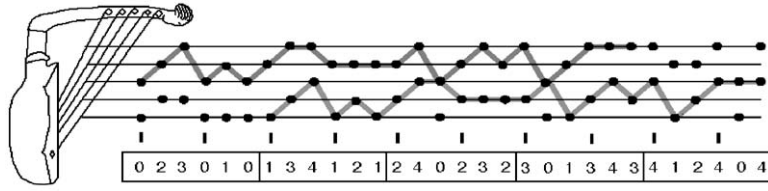


Fig. 1. A canon formula from African harp music.

$0 = (c_1, c_3)$, $1 = (c_1, c_4)$, $2 = (c_2, c_4)$, $3 = (c_2, c_5)$ and $4 = (c_3, c_5)$. A mapping t is defined by $t(c_5) = c_3$, $t(c_4) = c_2$, $t(c_3) = c_1$ from upper strings plucked by one hand to lower strings plucked by the other. In this case, the value of p is 6, and equality (3.1) defining a melodic canon can thus be written

$$u = t(v) \parallel T^6 v.$$

This is equivalent to

$$u(n) = (t(v(n)), v(n+6)) \quad (3.2)$$

for any integer n .

The main idea in the construction of melodic canons is to decompose these canons into independent infinite subwords computed by an automaton. To express this decomposition, we define the *homothetic* operator H_p , which associates to each infinite word u the infinite word $H_p u$ defined by

$$H_p u(n) = u(pn)$$

for any integer n .

For two infinite words u and v , it is obvious that $u = v$ if and only if $H_p T^i u = H_p T^i v$ for any i such that $0 \leq i < p$, since the set of integers $\mathbb{N}$ is equal to the union of the sets $i + p\mathbb{N}$ for $0 \leq i < p$. One has the following simple properties. We make an abuse of notation so as to avoid complicated notations, by using the same notation T and H_p whatever being the alphabet B , $E_0, \dots$, or E_{k-1} .

Proposition 3.1. (i) $H_p H_q = H_{pq}$,
(ii) T and t_j commute,
(iii) H_p and t_j commute,
(iv) $TH_p = H_p T^p$, which gives $T^l H_p = H_p T^{lp}$ for any l ,
(v) T and H_p preserve the superimposition of infinite words

$$T \left(\bigparallel_{0 \leq j \leq k-1} v_j \right) = \bigparallel_{0 \leq j \leq k-1} T(v_j) \quad H_p \left(\bigparallel_{0 \leq j \leq k-1} v_j \right) = \bigparallel_{0 \leq j \leq k-1} H_p(v_j).$$

We define an automaton R with labels in B , as a subgraph of the de Bruijn graph of order $k-1$. The set of vertices is the same, that is the set of finite words over B of

length $k - 1$, but some transitions are deleted. More precisely, there exists a transition in R labeled by a from $x_0 \dots x_{k-2}$ to $x_1 \dots x_{k-2}a$ if and only if the k -tuple a satisfies

$$a \in \{t_0(x_0^+)\} \times \{t_1(x_1^+)\} \cdots \times \{t_{k-2}(x_{k-2}^+)\} \times E_{k-1}.$$

Proposition 3.2. *An infinite word u is a k -voice melodic canon with distance p if and only if all infinite words $H_p T^i u$ for $0 \leq i < p$ are labels of infinite paths in the automaton R .*

Proof. The definition of a melodic canon (3.1), and equalities (iv) and (v) of Proposition 3.1, imply that

$$H_p u = \bigparallel_{0 \leq j \leq k-1} H_p T^{jp} t_j(v) = \bigparallel_{0 \leq j \leq k-1} T^j t_j(H_p v)$$

and more generally

$$H_p T^i u = \bigparallel_{0 \leq j \leq k-1} T^j t_j(H_p T^i v)$$

which means that the subwords $H_p T^i u$ are k -voice canons with distance 1. Conversely, if $H_p T^i u$ is a k -voice canon with distance 1 for any i such that $0 \leq i < p$, then u is a k -voice canon with distance p .

Consider now an infinite word u , which is a k -voice canon with distance 1. The equality

$$u = \bigparallel_{0 \leq j \leq k-1} T^j t_j(v)$$

can be expressed in terms of k -tuples of events

$$u(n) = (t_0(v(n)), t_1(v(n+1)), \dots, t_{k-2}(v(n+k-2)), v(n+k-1))$$

for any integer n . It shows that u is a k -voice canon with distance 1 if and only if u is the label of an infinite path in the automaton R . $\square$

Up to the end of this section, we consider the particular case of periodic infinite words. If m is the period of a k -voice canon, and p its distance, m and p may not be coprime. Thus we put $d = \gcd(m, p)$ with $p = p'd$, $m = m'd$, and m' and p' coprime. The subwords $H_d T^i u$ are called the *cyclic components* of u for $0 \leq i < d$. It is easy to show that these words are m' -periodic. One has the following property.

Proposition 3.3. *An infinite word u is a k -voice melodic canon with period m and distance p if and only if its cyclic components $H_d T^i u$ for $0 \leq i < d$ are k -voice melodic canons with period m' and distance p' .*

Proof. Since $p = p'd$, equality (3.1) may be written as

$$u = \bigparallel_{0 \leq j \leq k-1} T^{jp'd} t_j(v)$$

thus

$$T^i u = \big\|_{0 \leq j \leq k-1} T^{jp'd} t_j(T^i v)$$

which gives

$$H_d T^i u = \big\|_{0 \leq j \leq k-1} H_d T^{jp'd} t_j(T^i v) = \big\|_{0 \leq j \leq k-1} T^{jp'} t_j(H_d T^i v). \quad \square$$

The study of melodic canons is thus reduced to the study of their cyclic components. We can then assume that the period m and the distance p are coprime. Proposition 3.2 may be written in the following way.

Proposition 3.4. *Let m and p be coprime, and u an infinite word with period m . Then u is a k -voice melodic canon with distance p , if and only if the infinite subword $H_p u$ is the label of a cyclic path of length m in the automaton R .*

Proof. From Proposition 3.2, one can deduce that $H_p u$ is the label of an infinite path in the automaton R . Since R is a subgraph of the de Bruijn graph of order $k-1$, the vertices of this path are factors of length $k-1$ of $H_p u$ except the first $k-1$ vertices. But $H_p u$ being m -periodic, its m -th factor of length $k-1$ is equal to its first factor of the same length. The path is thus a cycle of length m , except its first $k-1$ vertices. Furthermore, the fact that $H_p u$ is periodic implies that its prefix of length $k-2$ can also be read around the cycle. Thus the whole infinite word $H_p u$ is the label of a cyclic path of length m in R .

Conversely, if $H_p u$ is the label of a cyclic path in the automaton R , one has to prove that the same holds for $H_p T^i u$ for $0 \leq i < p$. As we shall see, these words correspond to the same cycle. In fact, since m and p are coprime, p is a generator of $\mathbb{Z}/m\mathbb{Z}$. Thus there exists an integer $q < m$ such that $i \equiv p \cdot q$ modulo m . One has

$$H_p T^i u(n) = T^i u(pn) = u(pn + i) = u(pn + pq + mj),$$

and since u is m -periodic

$$H_p T^i u(n) = u(p(n + q)) = H_p u(n + q) = T^q H_p u(n),$$

which proves that $H_p T^i u = T^q H_p u$. Then if $H_p u$ is the label of a cyclic path in the automaton R , the same holds for $T^q H_p u$. $\square$

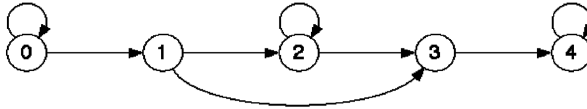
In the Nzakara case, some errors regarding the canon structure appear in the sequence represented in Fig. 1. A few pairs of notes have their lower note outside the lower broken line (exactly six pairs as one can verify by continuing the broken line). More formally, there exist some values of n for which the relation (3.2) defining the canon is not verified. Propositions 3.3 and 3.4 lead to a corollary, which proves that it is not possible to construct a strict two-voice canon in the Nzakara case.

We say that a cyclic component is *trivial* if it is a constant sequence, taking as value a unique element of B . We extend the definition of a melodic canon, in order

to include the case where there exist some values of n , called *errors*, such that the relation (3.2) above, $u(n) = (t(u(n)^+), u(n+6)^+)$, is not satisfied.

Corollary 3.5. *In the Nzakara case, any melodic canon with period m and distance p , which has no trivial cyclic component, has at least $d = \gcd(m, p)$ errors.*

Proof. In the Nzakara case, one has $k=2$. The automaton R has edges from x to y labeled by y (labels on the edges are omitted) if and only if the first component of y is $t(x^+)$.



As one can see, there is no cyclic path in the graph of R except paths with a single edge. It follows that if u has no trivial cyclic component, each component $H_d T^i u$ for $0 \leq i < d$ contains at least one error. Thus u must contain at least d errors. $\square$

Example. The word corresponding to the harp sequence represented in Fig. 1 has the following six cyclic components, with $m=30$, $p=6$ and $d=6$,

$$\begin{aligned} u &= 0\ 2\ 3\ 0\ 1\ 0\ 1\ 3\ 4\ 1\ 2\ 1\ 2\ 4\ 0\ 2\ 3\ 2\ 3\ 0\ 1\ 3\ 4\ 3\ 4\ 1\ 2\ 4\ 0\ 4, \\ H_6 u &= 0\ 1\ 2\ 3\ 4, \\ H_6 T u &= 2\ 3\ 4\ 0\ 1, \\ H_6 T^2 u &= 3\ 4\ 0\ 1\ 2, \\ H_6 T^3 u &= 0\ 1\ 2\ 3\ 4, \\ H_6 T^4 u &= 1\ 2\ 3\ 4\ 0, \\ H_6 T^5 u &= 0\ 1\ 2\ 3\ 4. \end{aligned}$$

As one can see, these cyclic components are cyclic shifts of the same sequence 0 1 2 3 4. They could be considered as labels of cyclic paths in the automaton R represented in the proof of Corollary 3.5, provided an additional edge is traced from 4 to 0. Thus each cyclic component contains exactly one error, corresponding to this missing edge, and the harp formula contains six errors. This is a general property of Nzakara harp canons, which always contain the minimal number of errors indicated by Corollary 3.5. They have an additional property, since their cyclic components are labels of the same cyclic path of R . Thus one can exchange rows and columns in the definition of these components, and this transformation gives the following five words v_0 , v_1 , v_2 , v_3 , and v_4 each of them being obtained from the previous one by adding the same value to its elements (1 added to 0 2 3 0 1 0 gives 1 3 4 1 2 1, and so on), until we reach the initial word again

$$\begin{aligned} v_0 &= 0\ 2\ 3\ 0\ 1\ 0, \\ v_1 &= 1\ 3\ 4\ 1\ 2\ 1 = v_0 + 1, \end{aligned}$$

$$v_2 = 2\ 4\ 0\ 2\ 3\ 2 = v_1 + 1,$$

$$v_3 = 3\ 0\ 1\ 3\ 4\ 3 = v_2 + 1,$$

$$v_4 = 4\ 1\ 2\ 4\ 0\ 4 = v_3 + 1.$$

This appears to be a general construction in the Nzakara harp repertoire. The fact that all cyclic components have the same cyclic path in the automaton R is a particular case of the construction expressed by Proposition 3.4 for building melodic canons. Next section is devoted to the study of this case.

4. Redundancy and finite difference calculus

In this section, we study a particular case of periodic melodic canons appearing in the Nzakara repertoire. The cyclic components of these particular melodic canons are cyclic shifts of one another. As we have seen in the previous example, the construction of such canons relies on the translation of a given sequence by adding the same value to its elements until the initial sequence is reached again.

This section deals with the following enumeration problem: we want to compute all the canons that can be made in this way, depending on the given sequence and the given translation value. The problem is that an infinite periodic word prefix of another one is considered as the same musical sequence (recall that harp formula are repeated as a loop). Thus we want to restrict ourselves to solutions not prefix of another. This means that we want to compute a cross-section of the set of translated infinite words with regard to the conjugacy relation. This can be done by introducing Lyndon words and the finite difference calculus studied by Vuza [2].

Up to the end of this section, we assume that the alphabet B is a finite Abelian group G . In the Nzakara case, B is the set of the five pairs of strings denoted by 0, 1, 2, 3, 4, so one can identify B with $\mathbb{Z}/5\mathbb{Z}$. We define the *difference word* Du of an infinite word u by

$$Du(n) = u(n+1) - u(n)$$

for any integer n . The set of infinite words over G is an Abelian group. The operator D is a morphism, and the kernel of D is the subgroup $\text{Ker}(D)$ of constant infinite words, that is to say 1-periodic words taking only one single value in G . Notice that the product of T and D is commutative, where T is the translation operator

$$DT = TD.$$

One has the following basic property.

Proposition 4.1. (i) *If $Du = Dv$, then $u - v$ is a constant infinite word.*

(ii) *If u is m -periodic, then Du is m -periodic. Conversely, if Du is m -periodic, then u is km -periodic for some $k \geq 1$ (where k is the order of an element of G).*

Proof. (ii) Assume $T^m Du = Du$. One has $DT^m u = Du$, thus $T^m u = u + z$ with z in G . Let k be the order of z in the group G . Then $T^{km} u = u + kz = u$. $\square$

We shall restrict ourselves to periodic infinite words, and we denote by G_m the finite group of infinite m -periodic words, which means words having a period which is less than m and which divides it. The kernels and images of D^k for $k \geq 0$ make a pair of sequences of included sets, an increasing one for the kernels, and a decreasing one for the images. It can be shown that these two sequences become stationary for the same value of k , for which the intersection of $\text{Ker}(D^k)$ and $\text{Im}(D^k)$ is reduced to $\{0\}$. Since there exists a bijection between $G_m/\text{Ker}(D^k)$ and $\text{Im}(D^k)$, a cardinality argument concerning the direct sum of $\text{Ker}(D^k)$ and $\text{Im}(D^k)$ leads to the following equality

$$G_m = \text{Ker}(D^k) \oplus \text{Im}(D^k). \quad (4.1)$$

One can improve slightly this result, since the restriction of D^k to $\text{Im}(D^k)$ is a bijection, and the group of bijections of $\text{Im}(D^k)$ is finite, so that there exists an integer l such that the restriction of $(D^k)^l$ to $\text{Im}(D^k)$ is the identity. Replacing k by kl in the direct sum (4.1) above proves that every infinite m -periodic word u can be decomposed in a unique way $u = f + g$, where f is a *reducible* part satisfying $D^k f = 0$, and g a *reproducible* part satisfying $D^k g = g$ [2].

We say that a periodic infinite word over G is *redundant* if the period of Du is strictly less than the period of u . Note that in this case, if r and m are the periods of Du and u respectively, then r must divide m , so that there exists an integer q called the *rate of redundancy* such that $m = rq$. An infinite periodic word, which is redundant, cannot be reproducible, and this implies that its reducible part cannot be equal to zero.

Example. The harp sequence shown in Fig. 1

$$u = 0 \ 2 \ 3 \ 0 \ 1 \ 0 \ 1 \ 3 \ 4 \ 1 \ 2 \ 1 \ 2 \ 4 \ 0 \ 2 \ 3 \ 2 \ 3 \ 0 \ 1 \ 3 \ 4 \ 3 \ 4 \ 1 \ 2 \ 4 \ 0 \ 4$$

is a redundant word since u has period $m = 30$, and Du has period $r = 6$

$$Du = 2 \ 1 \ 2 \ 1 \ 4 \ 1.$$

As expected, the reducible part of u is not equal to zero, since the decomposition $u = f + g$ into reducible and reproducible parts gives

$$f = 1 \ 2 \ 3 \ 4 \ 0,$$

$$g = 4 \ 0 \ 0 \ 1 \ 1 \ 4.$$

One can easily verify that f is reducible since $D^2 f = 0$, and g is reproducible since $D^8 T^2 g = g$, which implies that $(D^8 T^2)^3 g = g$, thus $D^{24} T^6 g = g$, and since the period of g is equal to 6, one has $D^{24} g = g$.

Proposition 4.2. *If u is a redundant word, r the period of Du , and q the rate of redundancy, then $T^r u = u + z$ where z is an element of order q in G . Conversely, if u and $u + z$ are conjugate for some non-zero element z in G , then u is redundant.*

Proof. We have $T^r Du = Du$ (this is the hypothesis on the period of Du). Since D and T commute, we have $DT^r u = Du$. Then we may conclude by Proposition 4.1. (i) that $T^r u = u + z$ with z in G . Since u has period rq , then $u = T^{rq} u = u + qz$. Thus $qz = 0$.

Conversely, assume $T^r u = u + z$. Then $T^r Du = DT^r u = D(u + z) = Du$, so that Du is r -periodic. But since z is not equal to zero, u is not r -periodic. Thus u is redundant. $\square$

Corollary 4.3. *If the set of redundant words with a rate of redundancy equal to q is not empty, then q divides the order of the group G .*

Let us denote by $R(r, z)$ the set of all redundant infinite words u such that r is the period of Du , and z the element of G such that $u(n + r) = u(n) + z$. The following property has a converse part, which holds only in a very special case. The interesting fact is that Nzakara harp formulas precisely fall into this case.

Proposition 4.4. *If u and v are conjugate, then Du and Dv are conjugate. Conversely, if Du and Dv are conjugate, and v is a redundant word of $R(r, z)$ with the order of z being equal to $\text{card}(G)$, then u and v are conjugate.*

Proof. For the converse part, let $n = \text{card}(G)$, and $T^r v = v + z$ where z has order n . Assume Du and Dv are conjugate. Then $Dv = T^k Du = DT^k u$, so that $v = T^k u + y$ with y in G . One has $T^{ir} v = v + iz$, so that $T^{k+ir} u + y = v + iz$. Since z has order n , there exists i such that $iz = y$. This gives $T^{k+ir} u = v$, which proves that u and v are conjugate. $\square$

In the Nzakara case, the rate of redundancy always has value $q = 5$, which is the order of the group $\mathbb{Z}/5\mathbb{Z}$. Thus Proposition 4.4 gives an efficient algorithm to compute a cross-section of the set $R(r, z)$ for the conjugacy relation, given the values of r and z . Indeed, one has the following simple set theoretic proposition.

Proposition 4.5. *Let f be a mapping from E to F , $\sim_E$ and $\sim_F$ equivalence relations on E and F respectively, and W a subset of E . If for any $x, y \in W$, $f(x) \sim_F f(y)$ is equivalent to $x \sim_E y$, then there is a bijection between $W/\sim_E$ and $f(W)/\sim_F$.*

Proof. We may define f' from $W/\sim_E$ to $f(W)/\sim_F$ by $f'(c_x) = c_{f(x)}$ where c_x is the equivalence class of x . It is possible since for any $y \in c_x \cap W$, one has $c_{f(y)} = c_{f(x)}$ because $x \sim_E y$ implies $f(x) \sim_F f(y)$. Then $f'(c_x) = f'(c_y)$ means $f(x) \sim_F f(y)$, which implies by hypothesis $x \sim_E y$, whence $c_x = c_y$. Thus f' is injective. $\square$

Algorithm. Considering the mapping D from G_m to G_m and applying Proposition 4.5 to the converse part of Proposition 4.4, with $W = R(r, z)$, one gets a bijection between a cross-section of $D(R(r, z))$ and a cross-section of $R(r, z)$ for the conjugacy relation. The advantage is that words in $D(R(r, z))$ have a much shorter period than those in $R(r, z)$. For instance, in the example above, one has $r = 6$, so that words in $D(R(6, z))$ have period 6, whereas words in $R(6, z)$ have period 30. As we have recalled in Section 2, the computation of a cross-section for the conjugacy relation relies on Lyndon words. Thus we have to compute Lyndon words of length r instead of length $m = rq$, which gives a much faster computation. This method based on Proposition 4.5 seems to be a general technique for the enumeration of musical circular structures (see [16]).

Example. The following musical word, appearing in the Nzakara harp repertoire, has a remarkable property of uniqueness

$$w = 0\ 1\ 3\ 4\ 1\ 2\ 4\ 0\ 2\ 3.$$

It is obtained from the initial sequence of two elements 0 1 (one has $r=2$), with the translating value $z=3$. In this case, the problem of enumerating all the sequences obtained by translating an initial sequence of length 2 may be resolved easily by hand, since one can fix the first element of the initial sequence to 0, and there are only five possible values for its second element, as shown in the list below

$$w_0 = 0\ 0\ 3\ 3\ 1\ 1\ 4\ 4\ 2\ 2,$$

$$w_1 = 0\ 1\ 3\ 4\ 1\ 2\ 4\ 0\ 2\ 3 = w,$$

$$w_2 = 0\ 2\ 3\ 0\ 1\ 3\ 4\ 1\ 2\ 4 = T^8 w,$$

$$w_3 = 0\ 3\ 3\ 1\ 1\ 4\ 4\ 2\ 2\ 0,$$

$$w_4 = 0\ 4\ 3\ 2\ 1\ 0\ 4\ 3\ 2\ 1.$$

Among the resulting sequences, w_0 and w_3 are not acceptable as Nzakara harp formulas, since they have repeated elements. The sequence w_4 is a kind of “degenerated” sequence, since it is split into two equal shorter sequences. And the remarkable fact is that the two remaining sequences w_1 and w_2 are both cyclic shifts of one another, so that in this case, the Nzakara harp sequence w is the only solution to the problem. More generally, the problem of enumerating Nzakara harp canons has few solutions with the minimal number of errors.

Remark. The notion of “redundant infinite word” introduced in this section has another interesting musical application, in the characterization of Olivier Messiaen’s *modes of limited transposition*. In this case, the alphabet is the group $\mathbb{Z}/12\mathbb{Z}$ corresponding to the twelve chromatic notes. The following mode C D Eb E F# G Ab Bb B (which is Messiaen’s third mode) may be written as

$$u = 0\ 2\ 3\ 4\ 6\ 7\ 8\ 10\ 11.$$

The modes of limited transposition are defined by the property that an element z of $\mathbb{Z}/12\mathbb{Z}$ exists such that u and $u+z$ are conjugate (in fact, one has in this case $u+4=T^3u$). As a consequence of Proposition 4.3, u is redundant, as it can easily be verified since its period is 9, whereas the difference word Du associated with u has period 3:

$$Du = 2\ 1\ 1.$$

The computation of all modes of limited transposition is done by various authors (for instance [25,30]).

[illegible]

is then equivalent to finding polynomials P_1 and P_2 such that

$$J(x)P_1(x) + J(x_2)P_2(x) = I_{15}(x),$$

where I_n is the polynomial of degree n with all coefficients being equal to 1. This elegant formalization has been used by Emmanuel Amiot to prove Tom Johnson's conjecture: *every rhythmic augmented canons based on pattern 1 1 0 0 1 and covering the time line has a length that is a multiple of 15* [1,20].

For languages X and Y of A^* , we define the *superimposition* of X and Y as

$$X \parallel Y = \{u \parallel v, u \in X, v \in Y\}.$$

Then we denote $X^{(0)} = \{1\}$, and $X^{(n+1)} = X^{(n)} \parallel X$ for any integer n . The *iterated superimposition* of X is defined by

$$X^\circ = \bigcup_{n \geq 0} X^{(n)} = 1 \cup X \cup (X \parallel X) \cup (X \parallel X \parallel X) \dots$$

This section shows how to formalize the basic rule of serial music by means of the iterated superimposition.

Example. The example shown in Fig. 2 is called a “serial” sequence, because it can be analyzed as two occurrences of the same series of twelve notes, indicated by numbers from 1 to 12 added to the score.

More formally, let us denote this passage as a sequence of simultaneous events

$$z = 1 \ 2 \ 3 \ 4 \ 5 \ 6 \ 7 \ 8 \ 9 \ 10 \ 11 \ \{1,2\} \ \{3,4\} \ 12 \ \{5,6\} \ 7 \ 8 \ 9 \ 10 \ 11 \ 12.$$

The two occurrences of the series can be represented as two sequences v_1 and v_2 with

$$\begin{aligned} v_1 &= 1 \ 2 \ 3 \ 4 \ 5 \ 6 \ 7 \ 8 \ 9 \ 10 \ 11 \ \emptyset \ \emptyset \ 12, \\ v_2 &= \emptyset \ \emptyset \ \emptyset \ \emptyset \ \emptyset \ \emptyset \ \emptyset \ \emptyset \ \emptyset \ \emptyset \ \{1,2\} \ \{3,4\} \ \emptyset \ \{5,6\} \ 7 \ 8 \ 9 \ 10 \ 11 \ 12. \end{aligned}$$

The fact that z is serial can be expressed by the existence of a decomposition of z into $z = v_1 \parallel v_2$.

Let E be the set of the twelve chromatic notes, and A the power set of E . For any word w in A^* or E^* , we denote by $eve(w)$ the set of notes occurring in w . A *series* is a word u of E^* such that each note of E appears in u exactly once. For a given series u , we define $F_0(u)$ as the set of words v in A^* such that $v = eve(u_0)eve(u_1) \dots eve(u_k)$ where $u = u_0u_1 \dots u_k$ is a factorization of the series. Thus letters of v are sets of consecutive notes of the series.

We denote by π the projection from A^* to $(A \setminus \{\emptyset\})^*$ defined by $\pi(\emptyset) = 1$, and $\pi(a) = a$ for every non-empty letter, and we introduce the set $F(u)$ of *serial forms* of u

$$F(u) = \pi^{-1}(F_0(u)).$$

Fig. 2. Schoenberg's *Valse* for piano op. 23 (meas. 29 to 33).

A musical word z of A^* is said to be *serial* according to the series u if there exist serial forms $v_1, v_2, \dots, v_n$ in $F(u)$ such that

$$z = v_1 \parallel v_2 \dots \parallel v_n. \quad (5.1)$$

The language of serial words of A^* according to the series u , denoted by L , may thus be written as the iterated superimposition of $F(u)$

$$L = F(u)^\circ.$$

Our definition of serial sequences leads to a simple characterization of sequences that are serial according to a given series.

Proposition 5.1. *A word w is serial according to a series u if and only if every note x occurring in w is preceded (respectively followed) by the notes preceding x in u (respectively following x).*

Proof. The condition states that for every a in A such that $w = ras$, and every note x in a such that $u = u'xu''$, one has $eve(ra) \supset eue(u')$ and $eue(as) \supset eue(u'')$.

The condition is obviously necessary, since it is verified for every serial form.

To prove it is sufficient, consider the twelve-note series $u = x_1x_2 \dots x_{12}$. Let a be an element of A occurring in w , with $w = ras$, and let y be a note in a . Denoting by x_i the note in u equal to y , and k the length of ra , we define $v(x_i) = \emptyset^{k-1}x_i$. One has $v(x_i) \leq ra$ regarding the order relation associated with the superimposition. Furthermore, since x_{i-1} precedes x_i in u , the condition asserts that x_{i-1} occurs in ra . Thus we define $v(x_{i-2})$ in $\emptyset^*x_{i-2}$ such that $|v(x_{i-2})| \leq |v(x_{i-1})|$. Iterating the process for every x_j down to x_1 , and applying the same argument for notes x_j following x_i in u , with $j \geq i$, we define

$$v(y) = \parallel_{1 \leq j \leq 12} v(x_j)$$

which is by construction an element of $F(u)$. Furthermore, it satisfies $v(y) \leq w$ regarding the order relation associated with the superimposition. Denoting by $w = w_1w_2 \dots w_n$

the letters in w , one can write

$$w = \bigvee_{j \leq n, y \in w_j} v(y)$$

which proves that w is an element of $F(u)^\circ$. $\square$

The serial language L associated with a given series u appears to be regular. This is a consequence of a result proved in the next section (Theorem 6.4), asserting that the iterated superimposition of a regular language is also regular. Since $F(u)$ is the inverse image of a finite subset by a morphism, it follows that it is a regular language, thus $L = F(u)^\circ$ is regular.

Unfortunately, the proof of Theorem 6.4 does not provide an explicit construction of the automaton recognizing L . Proposition 5.1 may be written in a negative way, which leads to a regular expression of the complement of L . Indeed, a word w is not serial if and only if there exists a pair of notes (x, y) such that x precedes y in the series, and that one of the following conditions is satisfied

- (i) x appears in w , and y does not appear after x , or
- (ii) y appears in w , and x does not appear before y .

Let us denote by A_x the subset of the elements of A containing the note x . The two previous conditions lead to the following corollary, which gives a regular expression for the complement of L .

Corollary 5.2. *The complement of L is the union of sets of the form*

$$A^*(A_x \cap \bar{A}_y)\bar{A}_y^* + \bar{A}_x^*(A_y \cap \bar{A}_x)A^*$$

for all pairs of notes (x, y) such that x precedes y in the series.

Remark. Corollary 5.2 can be used to implement a simple algorithm verifying that a passage of a musical sequence satisfies the decomposition (5.1) according to a given series (as it is shown in [14], which gives an implementation of the algorithm in Lisp). Notice that this problem is much easier than the problem of computing the unknown series associated with a piece considered as serial. But the resolution of this second problem would require a more precise definition of what is a serial piece than the decomposition into serial forms given above (5.1). This decomposition is only a necessary condition for a piece to be serial, but it is far from being sufficient, so that the actual serial language is just a small proper subset of L . To illustrate this, consider the fact that Proposition 5.1 implies that a sequence is serial as soon as an element of $F_0(u)$ is read at the beginning of the sequence, and another one at the end, whatever happens in the middle, so that $F_0(u)A^*F_0(u)$ is included in L .

6. Closure properties for rational languages

In this section, we study closure properties of rational languages under both the superimposition and the iterated superimposition. But it will be convenient to consider

a more general binary operation on A^* , denoted by $\perp$. We associate to this operation a transduction t from $A^* \times A^*$ to A^* defined by

$$t = \{(x, y, x \perp y), x, y \in A^*\}.$$

The simple but fundamental result below is taken from [19, p. 243].

Proposition 6.1. *Let $\perp$ be a binary operation defined on A^* , and t the associated transduction in $A^* \times A^* \times A^*$. If t is rational, then for every regular languages X and Y of A^* , $X \perp Y$ is also regular.*

Proof. One has $X \perp Y = t(X \times Y)$. As recalled in Section 2, the Cartesian product $X \times Y$ of two regular languages is a recognizable subset of $A^* \times A^*$ (Mezei's theorem). But t being rational, $t(X \times Y)$ is a rational subset of A^* , hence a regular language. $\square$

We now assume that the binary operation $\perp$ on A^* has the empty word 1 as its identity. Moreover, we assume that it satisfies a property, called the *double distributivity property*, which asserts that for every words u, v, w and z , the following equality

$$uv \perp wz = (u \perp w)(v \perp z)$$

holds provided $|u| = |w|$.

The main consequence of this property lies in the following result.

Proposition 6.2. *If a binary operation $\perp$ on A^* has the empty word 1 as its identity, and satisfies the double distributivity property, then the associated transduction $t = \{(x, y, x \perp y), x, y \in A^*\}$ is rational.*

Proof. Let $u = a_1 \dots a_p$ et $v = b_1 \dots b_q$. Assume that $q > p$. One has

$$\begin{aligned} u \perp v &= (a_1 \perp b_1)(a_2 \dots a_p \perp b_2 \dots b_q) \\ &= (a_1 \perp b_1) \dots (a_p \perp b_p)(1 \perp b_{p+1} \dots b_q) \\ &= (a_1 \perp b_1) \dots (a_p \perp b_p)b_{p+1} \dots b_q. \end{aligned}$$

One has a similar relation if $q < p$. We define three finite subsets

$$\begin{aligned} H &= \{(a, b, a \perp b), a, b \in A\}, \\ D &= \{(a, 1, a), a \in A\}, \\ G &= \{(1, a, a), a \in A\}. \end{aligned}$$

The transduction t may be written as the rational expression

$$t = H^*(D^* + G^*)$$

which proves it is rational. $\square$

Both the superimposition and the literal shuffle are binary operations on A^* satisfying the double distributivity property, thus being associated with a rational transduction of

$A^* \times A^* \times A^*$. Proposition 6.1 proves that these two operations preserve the rationality of languages. One can thus find a classical result asserting that the literal shuffle of two regular languages is also regular [4]. A corresponding result concerning the superimposition may be stated as follows.

Theorem 6.3. *If X and Y are regular languages, then the superimposition $X \parallel Y$ is also regular.*

Considering the iterated operation, as defined in Section 5, one can observe differences between the superimposition and the literal shuffle. The iterated literal shuffle of a regular language is not necessarily regular. Indeed, the iterated literal shuffle of $a^2(ab)^*b^2$, denoted by L , is not regular. The argument relies on the iteration lemma. The numbers of a and b are equal for words in L . But it is possible to construct a sequence of words of L with an arbitrary long suffix containing only b . If L is regular, then one could iterate a factor containing only b , contradicting the fact that the number of a is equal to the number of b [4].

By contrast, the iterated superimposition satisfies a closure property stated below as Theorem 6.4. A first proof of this result was given by Michel Latteux, and we give in this section a different proof establishing a more general result stated as Theorem 6.6. It relies greatly on the fact that the superimposition satisfies *associativity*, *commutativity*, and *idempotence* properties. None of these properties is satisfied by the literal shuffle, and this explains the differences between iterated superimposition and iterated literal shuffle.

Theorem 6.4. *If X is a regular language, then its iterated superimposition X° is also regular.*

We assume that the binary operation $\perp$ on A^* satisfies the following properties:

- (i) the empty words 1 is the identity,
- (ii) double distributivity,
- (iii) commutativity, associativity, idempotence.

We define the iterated operation of $\perp$ as in Section 5, and the language obtained by applying the resulting operation to X is denoted by X° , such that

$$X^\circ = 1 \cup X \cup (X \perp X) \cup (X \perp X \perp X) \dots$$

Lemma 6.5. (i) *If u and v are words with equal length, then $u \perp v$ has the same length.*

(ii) *If the length of r is less than the length of t , then $r \perp ty = (r \perp t)y$.*

(iii) *If $w_1y_1, \dots$ and w_ny_n are words such that all w_i have the same length, then*

$$\left(\bigperp_{i \leq n} w_i y_i \right) = \left(\bigperp_{i \leq n} w_i \right) \left(\bigperp_{i \leq n} y_i \right).$$

(iv) If W is a set of words with equal length, then

$$\left(\bigoplus_{wy \in WY} wy \right) = \left(\bigoplus_{w \in W} w \right) \left(\bigoplus_{y \in Y} y \right).$$

Proof. (i) This results from the fact that $a \perp b \in A$ if $a, b \in A$. Indeed, assume $a \perp b = cw$, then one has $cw = a \perp cw = (a \perp c)w$, thus $a \perp c = c$. In the same way $b \perp c = c$. Both give $(a \perp b) \perp c = c$, then $cw \perp c = c$, which gives $(c \perp c)w = c$, thus $w = 1$. Conversely, assume $a \perp b = 1$, it follows $a = (a \perp b) \perp a = a \perp b = 1$, which is not true.

(ii) Let $t = t't''$ with $|t'| = |r|$. One has $r \perp ty = (r \perp t')t''y = (r \perp t)y$.

(iii) The associativity and (i) show that one can apply $n - 1$ times the double distributivity property.

(iv) Let $p = \text{card}(W)$, $q = \text{card}(Y)$, and $n = pq$. The elements of WY are indexed by i from 1 to n , and denoted by $w_i y_i$, and then we apply (iii). Furthermore, the idempotence, commutativity and associativity of $\perp$ give

$$\left(\bigoplus_{i \leq n} w_i \right) = \left(\bigoplus_{w \in W} w \right)$$

and the same holds for Y . $\square$

This leads to the following result [9].

Theorem 6.6. *Let $\perp$ be a binary operation on A^* with 1 as its identity, associative, commutative, idempotent, and verifying the double distributivity. For any regular language X , the language X° is also regular.*

Proof. We show that there exists only a finite number of sets $u^{-1}X^\circ$. Let Q be the set of states of an automaton recognizing X , i its initial state, T the set of final states. If u is a prefix of length k of an element in X° , it is a prefix in the operation $\perp$ of shorter elements which can be grouped in a single element r and of larger elements w 's. We shall consider the states of the automaton which are reached when reading the words w 's. More precisely, if u is a word of A^* with length k , we denote by $\Sigma(u)$ the family of subsets S of Q such that there exists a set W of words of length k , and a word r in X° with length less than k satisfying the following conditions

$$\begin{aligned} iW &= S, \\ u &= r \perp \left(\bigoplus_{w \in W} w \right). \end{aligned}$$

The fact that z belongs to $u^{-1}X^\circ$ is equivalent to the fact that uz belongs to X° . This means that if u is the empty word (which is the case where $u^{-1}X^\circ = X^\circ$), there exist $v_1, \dots, v_n$ in X such that $uz = v_1 \perp \dots \perp v_n$. Let V_1 be the set of v_i with length less than k , and V_2 the set of v_i with length greater than k .

We decompose the words v of V_2 as $v = wy$, with $|w| = k$ and we denote by W and Y the two sets of words defined by this decomposition, such that V_2 is included in

WY . We denote by

$$w_u = \left(\bigoplus_{w \in W} w \right), \quad y_u = \left(\bigoplus_{y \in Y} y \right).$$

All the words in W having the same length k , one has $|w_u| = k$ thanks to Lemma 6.5(i). Then Lemma 6.5(iv) gives

$$\begin{aligned} \left(\bigoplus_{w, y \in V_2} wy \right) &= \left(\bigoplus_{w \in W} w \right) \left(\bigoplus_{y \in Y} y \right) \\ &= w_u y_u. \end{aligned}$$

The commutativity and associativity of $\perp$ allow to group the elements of the decomposition of uz . Depending on the emptiness of V_1 and V_2 , we have one of the following situations:

(i) there exists r_u in X° with length $\leq k$, and $w_u y_u$ in X° with $|w_u| = k$ such that $uz = r_u \perp w_u y_u$. Thanks to Lemma 6.5(ii), one has $uz = (r_u \perp w_u) y_u$. In this case, both V_1 and V_2 are not empty, and since $|u| = |r_u \perp w_u|$, one can write

$$\begin{aligned} u &= r_u \perp w_u, \\ z &= y_u. \end{aligned}$$

(ii) there exists r_u with the same properties as in (i), such that $uz = r_u$, which means that

$$\begin{aligned} u &= r_u, \\ z &= 1. \end{aligned}$$

(iii) there exists $w_u y_u$ with the same properties as in (i), such that $uz = w_u y_u$, which means that

$$\begin{aligned} u &= w_u, \\ z &= y_u. \end{aligned}$$

Conditions (i), (ii), (iii) show that if z belong to $u^{-1}X^\circ$, then there exists S in $\Sigma(u)$ such that z belongs to

$$\bigoplus_{q \in S} (q^{-1}T)^\circ \setminus \{1\}.$$

Conversely, we show that if z satisfies this condition, then z belongs to $u^{-1}X^\circ$. Let $q_1, \dots, q_m$ be elements of S , and for j from 1 to m , let Y_j be the subset of $q_j^{-1}T$ such that

$$z = \left(\bigoplus_{y \in Y_1} y \right) \perp \dots \perp \left(\bigoplus_{y \in Y_m} y \right).$$

Since S belongs to $\Sigma(u)$, let W such that $i.W = S$, and r in X° with length less than k . We denote by W_j the subset of W such that $i.W_j = q_j$. It follows $u = r \perp u'$ with

$$u' = \left(\bigoplus_{w \in W_1} w \right) \perp \dots \perp \left(\bigoplus_{w \in W_m} w \right).$$

Lemma 6.1(iii) gives

$$u'z = \left(\bigoplus_{w \in W_1} w \right) \left(\bigoplus_{y \in Y_1} y \right) \perp \dots \perp \left(\bigoplus_{w \in W_m} w \right) \left(\bigoplus_{y \in Y_m} y \right),$$

then Lemma 6.1(iv) gives, according to the idempotence of $\perp$

$$u'z = \left(\bigoplus_{W_1 Y_1} w y \right) \perp \dots \perp \left(\bigoplus_{W_k Y_k} w y \right).$$

Since each $W_j Y_j$ is included in X , this proves that $uz = (r \perp u')z = r \perp u'z$ belong to X° .

There is only a finite number of $\Sigma(u)$, which proves that there is a finite number of $u^{-1}X^\circ$. $\square$

7. Remarks on rational order relations

A binary operation $\perp$ on A^* which is *associative*, *commutative*, and *idempotent* defines a structure of sup-semilattice associated with an order relation. We denote by σ this order relation on A^* defined by $\sigma(u, v)$ if and only if $u \perp v = v$. The operation $\perp$ is the max associated with σ , so that for every u and v in A^* , $\max(u, v)$ exists with value $u \perp v$.

We say that an order relation on A^* is *rational* if it defines a rational transduction in $A^* \times A^*$. Denoting by Δ the set $\{(u, u), u \in A^*\}$, the *strict order* relation associated with σ is defined by $\sigma_0 = \sigma \setminus \Delta$. Notice that Δ is rational, so that the rationality of σ_0 implies the rationality of $\sigma = \sigma_0 \cup \Delta$, but the converse is not true. Notice also that an order relation on A^* is never recognizable, since $\sigma \cap \sigma^{-1} = \Delta$, and Δ is not recognizable.

Proposition 7.1. *The order relation associated with the superimposition is a rational transduction of $A^* \times A^*$.*

Proof. Recall that the alphabet A is the set of subsets of E , where E is a given set of musical events. Thus one can say that a letter a is “included” in a letter b . We define

$$H = \{(a, b), a, b \in A, b \supset a\}.$$

The order relation σ associated with the superimposition has the following rational expression

$$\sigma = H^*(1 \times A^*)$$

which proves that it is rational. $\square$

This section addresses the question whether a connection exists between the rationality of a semi-lattice operation, and the rationality of its associated order relation. The first property obviously implies the second one, and the converse is true in a particular case. But the equivalence of these two properties is an open problem in the general case.

Proposition 7.2. *If the semi-lattice operation t is a rational transduction of $A^* \times A^* \times A^*$, then the associated order relation σ is a rational transduction of $A^* \times A^*$.*

Proof. $\sigma(x, z)$ is equivalent to the existence of y such that $\max(x, y) = z$. Considering the projection π from $A^* \times A^* \times A^*$ to $A^* \times A^*$ which associates (x, y, z) with (x, z) , one can write $\sigma = \pi(t)$, where σ and t are considered as subsets of $A^* \times A^*$ and $A^* \times A^* \times A^*$ respectively. This proves that σ is rational. $\square$

Proposition 7.3. *If σ is a total order and a rational transduction of $A^* \times A^*$, then the associated max operation t is a rational transduction of $A^* \times A^* \times A^*$.*

Proof. The mapping q_1 from $A^* \times A^*$ to $A^* \times A^* \times A^*$ which maps (x, y) to (x, y, y) is a morphism, thus $t_1 = q_1(\sigma)$ is rational. In the same way, the mapping q_2 , which maps (x, y) to (y, x, y) proves that $t_2 = q_2(\sigma)$ is rational. The transduction t associated with the max is thus rational, since $t = t_1 \cup t_2$. $\square$

Proposition 7.4. *If an order relation σ is a rational transduction of $A^* \times A^*$, then the transduction μ which associates (x, y) to all z such that $z \geq x$ and $z \geq y$ is a rational transduction of $A^* \times A^* \times A^*$.*

Proof. One can write $\mu = \{(x, y, z), (x, z) \in \sigma, (y, z) \in \sigma\}$. Let $\sigma = (\varphi, \psi)(K)$ be the decomposition of σ given by Nivat's theorem, as recalled in Section 2. We define F the morphism from $Z^* \times Z^*$ to $A^* \times A^* \times A^*$ which maps (u, v) to $(\varphi(u), \varphi(v), \psi(u))$, and P the intersection of $K \times K$ with the canonical equivalence of ψ . It is easy to show that P is a rational transduction of $Z^* \times Z^*$. Thus $\mu = F(P)$ is rational. $\square$

Establishing the rationality of the max operation associated with σ is equivalent to selecting among the triples (x, y, z) of μ having the same values for x and y , a triple with the minimal value for z , in such a way that the set of selected triples is rational. Thus the max operation appears to be a cross-section of the rational subset μ of $A^* \times A^* \times A^*$ according to the canonical equivalence relation of the projection that maps (x, y, z) to (x, y) (recall that the canonical equivalence relation of f is the set

of (u, v) such that $f(u) = f(v)$). The rationality of the max operation means that one can find a rational cross-section of μ .

Remark. Equivalence relations, which are similar to order relations, also lead to a problem concerning the existence of rational cross-sections. Johnson has conjectured that every rational equivalence relation has a regular cross-section. He proved in [21] that $\mathbf{KerRatF} = \mathbf{RatEq} \cap C_1$, where C_1 denotes the set of equivalence relations satisfying this conjecture, $\mathbf{RatEq}$ the set of rational equivalence relations of $A^* \times A^*$, and $\mathbf{KerRatF}$ the set of canonical equivalence relations for rational functions. Thus he established that his conjecture is equivalent to a second one asserting that $\mathbf{RatEq} = \mathbf{KerRatF}$.

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