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Computation of words satisfying the "rhythmic oddity property" (after Simha Arom's works)

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Abstract

This paper addresses the problem of enumerating all words having a combinatoric property called "rhythmic oddity property". This enumeration is motivated by the fact that this property is satisfied by many rhythmic patterns used in traditional Central African music.

Keywords: Algorithms; Combinatorial problems; Lyndon words; Combinatorics on words; Music formalization

1. Introduction

In 1952, ethnomusicologist Constantin Brailoiu wrote a paper on the combinatorics of asymmetric rhythmic patterns entitled "Le rythme aksak" [3]. These patterns are combinations of durations equal to two or three units, such as the famous Turkish rhythm 2 2 2 3 (see [7]). The asymmetry lies in the fact that they are based on two different durations. In his paper, Brailoiu gave a table of 1884 distinct rhythms, enumerating all the combinations that can be made with up to nine successive two- or three-unit elements.

This paper is devoted to the enumeration of a particular type of African asymmetric patterns, satisfying the "rhythmic oddity property" discovered by Simha Arom [1]. In Aka Pygmies music one can find the following rhythmic pattern

3 2 2 2 2 3 2 2 2 2 2 2.

The rhythmic oddity property asserts that when placing the two- and three-unit elements of the sequence on a circle (thus expressing the fact that the pattern is played as a loop), *one cannot break the circle into two parts of equal length whatever the chosen breaking point*. There is always one unit lacking on one side.

The asymmetry of the pattern is to some extent intrinsic, in the sense that there exists no breaking point giving two parts of equal length. Every division of the pattern gives two unequal parts, "half minus one" on the one side, and "half plus one" on the other side. Note that the oddity property requires that the circle is divided into an even number of units, so that it is possible to find patterns of the "half minus

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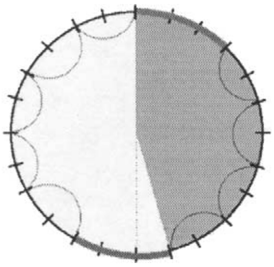

Fig. 1. No breaking point giving two parts of equal length.

one/half plus one" type. Many such patterns appear in Central African music, and this makes a challenging question of cognitive science, but we do not address this question here, since this paper is restricted to purely combinatorial aspects.

We describe an algorithm for enumerating all the patterns satisfying the rhythmic oddity property. The main idea of the construction is that patterns of this type must have an even number of three-unit elements, and that these elements must be placed nearly opposite around the circle. More precisely, if the units on the circle are numbered from 0 to $n-1$, a three-unit element beginning at i implies that a three-unit element begins either at $i+n/2-1$ or $i+n/2+1$ (modulo n). The construction is expressed in the paradigm of combinatorics on words, and we recall some basic notions from this domain.

2. The rhythmic oddity property

A word is a sequence of symbols from a given alphabet. In this paper, we consider words over the alphabet $A=\{2,3\}$.

We denote as usual by A^* the set of words over A, and by ε the empty word. For a word w, we denote by $|w|$ the length of w, and by $|w|_x$ the number of symbols equal to x in w. The concatenation of words is an associative operation and the empty word is a neutral element for concatenation. A word u is called a *prefix* (respectively *suffix*) of a word w if there exists a word y such that $w=uy$ (respectively $w=yu$).

In order to introduce the cyclic shifts of a word, let δ be the permutation of A^* defined by

$$\delta(\varepsilon)=\varepsilon, \qquad \delta(au)=ua, \quad a\in A,\ u\in A^*.$$

The *cyclic shifts* of w are the words of the form $\delta^k(w)$ for any integer k. For instance, the cyclic shifts of 2223 are 2223, 2232, 2322, and 3222.

The *height* of a word u, denoted by $h(u)$, is the sum of its symbols, and h is a morphism from A^* to $\mathbb{N}$.

A word w satisfies the *rhythmic oddity property* if and only if

(i) $h(w)$ is even, and
(ii) no cyclic shift of w can be factorized into words uv such that $h(u)=h(v)$.

Example. The condition $h(w)$ even is added to this definition, because if $h(w)$ is odd, the second condition is obviously satisfied for any word. In the Aka Pygmies example, one has $h(w)=24$, and the factorization of cyclic shifts gives

$$h(32222)=11, \quad h(322222)=13,$$
$$h(22223)=11, \quad h(222223)=13,$$
etc.

Proposition 1. *If w satisfies the rhythmic oddity property, then at least one of the following conditions is satisfied*:

(i) *there exists a unique pair (u,v) with $h(v)=h(u)+2$ such that $w=uv$, or*
(ii) *there exists a unique pair (u,v) with $h(v)=h(u)+2$ such that $w=vu$.*

Proof. First, the uniqueness of the factorizations of conditions (i) and (ii) is trivial. To prove the existence of such a factorization, let u be the longest prefix of w such that $h(u)<h(v)$, where v is the corresponding suffix with $w=uv$. We denote by x the first symbol of $v=xv'$. The two possible values for x are 2 and 3. One has $h(u)<h(v')+x$. Since u is maximal, one has $h(u)+x\geqslant h(v')$, but the rhythmic oddity property implies $h(u)+x>h(v')$, thus $|h(u)-h(v')|<x$. The possible values for x lead to the following three cases (i), (iia), (iib).

(i) If $x=2$, one has $w=u2v'$, and $h(w)$ being even implies that $|h(u)-h(v')|$ is even, thus equal to zero, and $h(v)=h(u)+2$ so that condition (i) of the property is satisfied by the pair (u,v), and condition (ii) is satisfied by the pair $(v',u2)$.

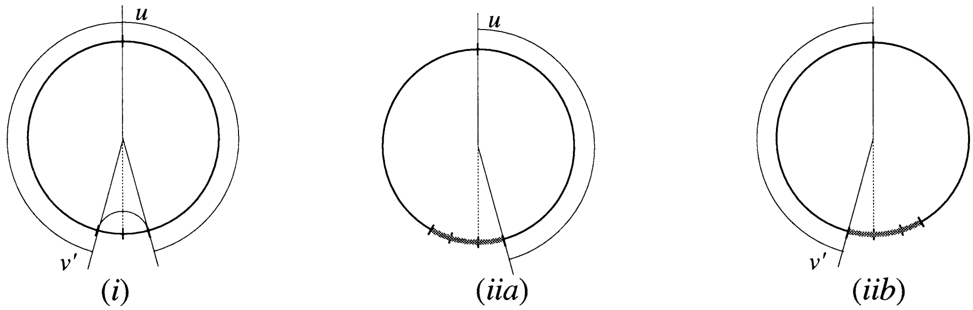

Fig. 2. Factorization of a word satisfying the rhythmic oddity property.

If $x = 3$, one has $w = u3v'$, and the fact that $h(w)$ is even implies that $|h(u) - h(v')|$ is odd, thus equal to one. This gives two remaining cases.

(iia) If $h(v') = h(u) - 1$, the equality $h(v) = h(v') + 3$ implies that $h(v) = h(u) + 2$ so that condition (i) of the property is satisfied by the pair (u, v).

(iib) If $h(v') = h(u) + 1$, the factorization $w = (u3)v'$ is such that $h(u3) = h(v') + 2$, so that condition (ii) of the property is satisfied by the pair $(v', u3)$. □

Proposition 1 implies that $2h(u) + 2 = h(w) = n$ (with n being even) so that $h(u) = n/2 - 1$ and $h(v) = n/2 + 1$, thus expressing the "half minus one/half plus one" characterization of these patterns given by Simha Arom [1].

We introduce the notion of asymmetric pair that will be the key of our construction. We say that (u, v) is an *asymmetric pair* if no pair of prefixes (u', v') of u and v respectively exist such that $h(v') = h(u') + 1$. For instance, (3322, 32322) is an asymmetric pair, but (3322, 32232) is not, since the pair of prefixes (33, 322) satisfies $h(322) = h(33) + 1$.

Proposition 2. *A word w satisfies the rhythmic oddity property if and only if there exists an asymmetric pair (u, v) such that $w = uv$ or $w = vu$ with $h(v) = h(u) + 2$.*

Proof. Let $w = uv$ or $w = vu$ with $h(v) = h(u) + 2$. The existence of a pair of prefixes (u', v') of u and v such that $u = u'u''$ and $v = v'v''$ with $h(v') = h(u') + 1$ is equivalent to the existence of a cyclic shift $v''u'u''v'$ of w (whatever being the factorization $w = uv$ or $w = vu$) such that $h(v''u') = h(u''v')$. Indeed, one has $h(v'') - h(u'') = h(u') - h(v') + 2$, which gives the following computation

$$h(v''u') = h(u''v') \Leftrightarrow$$
$$h(v'') - h(u'') = h(v') - h(u') \Leftrightarrow$$
$$h(u') - h(v') + 2 = h(v') - h(u') \Leftrightarrow$$
$$h(v') - h(u') = 1.$$

Thus if w satisfies the rhythmic oddity property, Proposition 1 gives $w = uv$ or $w = vu$ with $h(v) = h(u) + 2$. Then (u, v) must be an asymmetric pair. Conversely, if $w = uv$ or $w = vu$ where $h(v) = h(u) + 2$ and (u, v) being an asymmetric pair, then no cyclic shift of w can be factorized into words with equal height. □

3. Construction of asymmetric pairs

The following construction is inspired by Rauzy's rules, which are used to define standard pairs in the construction of characteristic Sturmian words [2]. Consider two functions a and b from $A^* \times A^*$ into itself defined by

$$a(u, v) = (3u, 3v), \qquad b(u, v) = (v, 2u).$$

One has the following proposition.

Proposition 3. *The set of asymmetric pairs is equal to the smallest set of pairs of words containing $\varepsilon \times A^*$ and $A^* \times \varepsilon$ and closed under a and b.*

Proof. Let F denote the smallest set of pairs of words containing $\varepsilon \times A^*$ and $A^* \times \varepsilon$ and closed under a and b, and P the set of asymmetric pairs. We prove $F = P$ by double inclusion. The first inclusion is obvious, since P contains $\varepsilon \times A^*$ and $A^* \times \varepsilon$ and is closed under a and b. The other inclusion is proved by induction on integer $|u| + |v|$, where (u, v) is an asymmetric pair.

The first case is $v = 2v'$. Then for each pair of prefixes (r, s) of u and v, one has $s = 2s'$ and (s', r) is a pair of prefixes of (v', u). If $h(r) = h(s') + 1$, then $h(s) = h(r) + 1$. This proves that if (u, v) is an asymmetric pair, then so is (v', u). By induction, (v', u) belongs to F, which proves that $(u, v) = b(v', u)$ belongs to F.

The other case is $v = 3v'$. Notice that 2 cannot be the first symbol of u, since (u, v) is an asymmetric pair, which implies that $(2, 3)$ cannot be a pair of prefixes of u and v. Thus $u = 3u'$. Then for each pair of prefixes (r, s) of u and v, one has $s = 3s'$ and $r = 3r'$ where (r', s') is a pair of prefixes of (u', v'). If $h(s') = h(r') + 1$, then $h(s) = h(r) + 1$. This proves that if (u, v) is an asymmetric pair, then so is (u', v'). By induction, (u', v') belongs to F, which proves that $(u, v) = a(u', v')$ belongs to F. □

Considering the free monoid B^* on the alphabet $B = \{a, b\}$, we identify the concatenation with the composition of functions. Thus any word α of B^* is identified with a function from $A^* \times A^*$ into itself. It is easy to prove by induction the following properties.

Proposition 4. *Let $(u, v) = \alpha(\varepsilon, \varepsilon)$ for a word $\alpha \in B^*$. If $|\alpha|_b$ is even, then $h(v) = h(u)$, and if $|\alpha|_b$ is odd, then $h(v) = h(u) + 2$.*

Proposition 5. *If $|\alpha|_b$ is even, then $\alpha(r, s) = \alpha(\varepsilon, \varepsilon).(r, s)$, and if $|\alpha|_b$ is odd, then $\alpha(r, s) = \alpha(\varepsilon, \varepsilon).(s, r)$.*

Corollary 6. *A word w satisfies the rhythmic oddity property if and only if there exists a word $\alpha \in B^*$ with $|\alpha|_b$, being odd, such that $w = uv$ or $w = vu$ with $(u, v) = \alpha(\varepsilon, \varepsilon)$.*

Let D be the subset of A^* defined by

$$D = \{w = uv, \exists\alpha \in B^*, |\alpha|_b \text{ odd}, \ (u, v) = \alpha(\varepsilon, \varepsilon)\}.$$

The set D does not include every words satisfying the rhythmic oddity property, but only part of them. Indeed, as expressed by Corollary 6, a word w satisfies the rhythmic oddity property if and only if w belongs to D, or w is a cyclic shift of a word that belongs to D ($w = vu$ with $uv \in D$).

For any $w \in D$, we put $f(w) = \alpha$, which is possible since α is unique. Indeed, following Proposition 1, a word w cannot have more than one factorization $w = uv$ with $h(v) = h(u) + 2$. Moreover, equality $(u, v) = \alpha(\varepsilon, \varepsilon) = \beta(\varepsilon, \varepsilon)$ for $\alpha, \beta \in B^*$ implies that $\alpha = \beta$ thanks to the fact that

$$\begin{aligned} &a(A^* \times A^*) \cap b(A^* \times A^*) \\ &\quad = (3A^* \times 3A^*) \cap (A^* \times 2A^*) = \emptyset. \end{aligned}$$

Thus f is a function from A^* to B^* with D as its domain. Moreover, the image $f(D)$ is the subset of words of B^* having an odd number of symbols equal to b, and f is injective, so that f is a bijection from D to $f(D)$.

Example. The computation of the image $f(w)$ of $w = 332232322$ proceeds as follows. First, the factorization of w into uv such that $h(v) = h(u) + 2$ gives $u = 3322$ and $v = 32322$. Then, one has

$$\begin{aligned} (u, v) &= (3322, 32322) \\ &= a(322, 2322) \\ &= ab(322, 322) \\ &= aba(22, 22) \\ &= abab(2, 22) \\ &= ababb(2, 2) \\ &= ababbb(\varepsilon, 2) \\ &= ababbbb(\varepsilon, \varepsilon), \end{aligned}$$

which gives $f(332232322) = ababbbb$.

Proposition 7. *For any $w, w' \in D$, $f(w')$ is a cyclic shift of $f(w)$ if and only if w' is a cyclic shift of w.*

Proof. Let $w \in D$ and $f(w) = \alpha$. First, we prove that α and $\delta(\alpha)$ are images of words that are cyclic shifts of one another. There are two cases $\alpha = a\alpha'$ and $\alpha = b\alpha'$ depending on the first symbol of α. We put $(u', v') = \alpha'(\varepsilon, \varepsilon)$.

In the first case, $a\alpha'(\varepsilon, \varepsilon) = (3u', 3v')$. Proposition 5 gives $\alpha' a(\varepsilon, \varepsilon) = \alpha'(3, 3) = \alpha'(\varepsilon, \varepsilon).(3, 3) = (u'3, v'3)$. Thus $a\alpha' = f(3u'3v')$ and $\alpha' a = f(u'3v'3)$ are images of cyclic shifts of one another.

In the second case, $b\alpha'(\varepsilon, \varepsilon) = (v', 2u')$. Since $|\alpha|_b$ is odd, $|\alpha'|_b$ must be even, so that Proposition 5 gives $\alpha' b(\varepsilon, \varepsilon) = \alpha'(\varepsilon, 2) = \alpha'(\varepsilon, \varepsilon).(\varepsilon, 2) = (u', v'2)$. As before, $b\alpha' = f(v'2u')$ and $\alpha' b = f(u'v'2)$ are images of cyclic shifts of one another.

For the converse part, the difficulty is that $\delta(w) \in D$ is not true for all $w \in D$. We shall prove that for $w \in D$ and $r > 0$ being the least integer such that $\delta^r(w) \in D$ (which exists since for $r = |w|$, one has $\delta^r(w) = w$), the images of w and $\delta^r(w)$ are cyclic shifts of one another. We put $w = uv$ with $(u, v) = \alpha(\varepsilon, \varepsilon)$. There are three cases (ia), (ib), (ii).

We first assume that w begins with 3, thus $w = 3u'v$. Since (u, v) is an asymmetric pair, $(3, 22)$ cannot be a pair of prefixes of u and v. Thus v begins with 3 or 23, which leads to the following two cases.

(ia) If $v = 3v'$, then $w = 3u'3v'$, which gives $f(w) = a\alpha'$. One has $\delta(w) = u'3v'3$ with $h(v'3) = h(u'3) + 2$. Proposition 5 gives $(u'3, v'3) = (u', v').(3, 3) = \alpha'(\varepsilon, \varepsilon).(3, 3) = \alpha'(3, 3) = \alpha' a(\varepsilon, \varepsilon)$, thus $\delta(w) \in D$ (so that in this case $r = 1$) with $f(\delta(w)) = \alpha' a$, which proves that the images of w and $\delta(w)$ are cyclic shifts of one another.

(ib) If $v = 23v'$, then $w = 3u'23v'$. Note that in this case, $\delta(w) = u'23v'3$ is factorized into words such that $h(v'3) = h(u'23) - 2$, so that $\delta(w)$ is not in D. Let $k - 1$ (respectively $l - 1$) be the length of the greatest prefix of u' with all symbols equal to 3 (respectively v'). One has $l \geqslant k$, because if $l < k$, then 3^l would be a prefix of u' whereas $3^{l-1}2$ is a prefix of v', thus $(33^l, 233^{l-1}2)$ would be a pair of prefixes of $3u'$ and $23v'$, respectively, which is not possible since $(3u', 23v')$ is an asymmetric pair. Thus $u' = 3^{k-1}2u''$ and $v' = 3^{k-1}v''$ with $h(v'') = h(u'') + 2$, and one has $w = 3^k 2u''23^k v''$. First, we have $\delta^s(w) \notin D$ for every $1 \leqslant s \leqslant k$. Indeed, $\delta^s(w) = 3^{k-s}2u''23^k v''3^s$ is factorized into words such that $h(3^{k-s}v''3^s) = h(3^{k-s}2u''23^s) - 2$. Then, one has $\delta^{k+1}(w) \in D$. Indeed, one has $\delta^{k+1}(w) = u''23^k v''3^k 2$. The image of w is $f(w) = ba^k b\alpha'$ with $\alpha'(\varepsilon, \varepsilon) = (u'', v'')$. Since $|\alpha'|_b$ is odd, Proposition 5 gives $(u''23^k, v''3^k 2) = (u'', v'').(23^k, 3^k 2) = \alpha'(\varepsilon, \varepsilon).(23^k, 3^k 2) = \alpha'(3^k 2, 23^k) = \alpha' ba^k b(\varepsilon, \varepsilon)$. It follows that $\delta^{k+1}(w) \in D$, so that we can put $r = k + 1$, and $f(\delta^r(w)) = \alpha' ba^k b$ is a cyclic shift of $f(w)$.

We now assume that w begins with 2, so that $w = 2u'v$. There is only one remaining case.

(ii) Since $(2, 3)$ cannot be a pair of prefixes of u and v, it follows that v begins with 2, and $w = 2u'2v'$. Thus $f(w) = bb\alpha'$ with $\alpha'(\varepsilon, \varepsilon) = (u', v')$. One has $\delta(w) = u'2v'2$, and Proposition 5 gives $(u'2, v'2) = (u', v').(2, 2) = \alpha'(\varepsilon, \varepsilon).(2, 2) = \alpha'(2, 2) = \alpha' bb(\varepsilon, \varepsilon)$, thus $\delta(w) \in D$ (so that in this case $r = 1$) with $f(\delta(w)) = \alpha' bb$, which proves that the images of w and $\delta(w)$ are cyclic shifts of one another. □

Example. Note that Proposition 7 is only true for words such that $|f(w)|_b = |f(w')|_b$ is odd. If $|f(w)|_b = |f(w')|_b$ is even, the proposition is false. Indeed, one has

$$bba(\varepsilon, \varepsilon) = bb(3, 3) = b(3, 23) = (23, 23),$$
$$bab(\varepsilon, \varepsilon) = ba(\varepsilon, 2) = b(3, 32) = (32, 23),$$

so that $bba = f(2323)$, $bab = f(3223)$, but 2323 and 3223 are not cyclic shifts of one another. On the contrary, one has

$$bbba(\varepsilon, \varepsilon) = b(23, 23) = (23, 223),$$
$$bbab(\varepsilon, \varepsilon) = b(32, 23) = (23, 232),$$

so that $bbba = f(23223)$, $bbab = f(32232)$, and 23223 and 32232 are cyclic shifts of one another.

The problem in the computation of words satisfying the rhythmic oddity property is that cyclic shifts of one another are considered as the same solution, since they correspond to rhythmic patterns repeated as a loop.

The *conjugacy* relation on words is the equivalence between words being cyclic shifts of one another, and the classes of conjugate elements are the orbits of the permutation δ. *Lyndon words* are defined as minimal elements in the conjugacy classes regarding the lexicographic order, with the additional condition that they are primitive (not power of another word). It follows that Lyndon words are canonic representatives for the conjugacy classes. Lyndon words can be computed efficiently in a recursive way thanks to the fact that Lyndon words not reduced to a single letter can be factorized into shorter Lyndon words lm such that $l < m$ for the lexicographic order [8]. For instance, the Lyndon word $aababb$ can be factorized into $(aab)(abb)$. Thus Lyndon words of length n are obtained by concatenating Lyndon words of length p and q with $n = pq$.

A naive algorithm for computing words satisfying the rhythmic oddity property up to a cyclic shift, would consists in computing Lyndon words on the alphabet $A = \{2, 3\}$, and deleting those which do not satisfy the property. This algorithm can be improved because a cross-section of D for the conjugacy relation is a cross-section of the set of words satisfying the rhythmic oddity property. Indeed, as we have said before, any word satisfying the rhythmic oddity property is either an element of D, or a cyclic shift of an element of D. The advantage of this remark is that a cross-section of D can be computed efficiently using the mapping f thanks to the following simple set theoretic Proposition 8.

Proposition 8. *Let f be a mapping from E to F, $\sim_E$ and $\sim_F$ equivalence relations on E and F, respectively, and D a subset of E. If for any $x, y \in D$, $f(x) \sim_F f(y)$ is equivalent to $x \sim_E y$, there is a bijection between $D/{\sim_E}$ and $f(D)/{\sim_F}$.*

Proof. We define f' from $D/{\sim_E}$ to $f(D)/{\sim_F}$ by $f'(c_x) = c_{f(x)}$, where c_x is the equivalence class of x. It is possible since for any $y \in c_x \cap D$, one has $c_{f(y)} = c_{f(x)}$ because $x \sim_E y$ implies $f(x) \sim_F f(y)$. Then $f'(c_x) = f'(c_y)$ means $f(x) \sim_F f(y)$, which implies by hypothesis $x \sim_E y$, whence $c_x = c_y$. Thus f' is injective. □

Considering the function f from A^* to B^*, Proposition 8 may be applied to the set D. It proves that there exists a bijection between a cross-section of D and a cross-section of $f(D)$ for the conjugacy relation. We have seen that the image $f(D)$ is the set of words of B^* having an odd number of symbols equal to b. It follows that the computation of a cross-section of $f(D)$ can easily be made using Lyndon words. In this way, we obtain a cross-section of D.

Furthermore, every conjugacy classes of words satisfying the rhythmic oddity property contains at least one element of D, as already mentioned. Finally, the computation of words satisfying the rhythmic oddity property is reduced to the computation of Lyndon words of B^* having an odd number of symbols equal to b.

Remark. Lyndon words provide a powerful tool for enumerating repetitive musical structures. It is sometimes convenient to replace A^* by another set $f(A^*)$ with an *ad hoc* mapping f. In our situation, the computation of Lyndon words is faster on the alphabet $B = \{a, b\}$ than on the alphabet $A = \{2, 3\}$, because the length of words of $f(D)$ is strictly less than the length of the corresponding words of D. For instance, considering

$$w = 332232322, \quad f(w) = ababbbb,$$

one has $|w| = 9$, whereas $|f(w)| = 7$. It seems that the method involving an *ad hoc* mapping f satisfying Proposition 8 is a general technique, which applies to different musical situations. We have encountered a similar example in [6], where we computed circular melodic canons.

4. Counting the solutions

Let n_2 and n_3 be the number of two- and three-unit elements of a solution. One has Table 1, with n_2 on the horizontal axis, and n_3 on the vertical one. We obtained these values experimentally by a constraint-based program [4], and then they were confirmed by an ILOG solver program designed by Louis-Martin Rousseau from the University of Montreal.

Let $X(p)$ be the number of words w satisfying the rhythmic oddity property up to a cyclic shift, where p is the length of $f(w)$. Since n_3 is even, we put $n_3 = 2j$ and $p = n_2 + j$, where j is the number of letters equal to a in $f(w)$.

Proposition 9. *If $n_3 = 2j$ is a power of 2, one has*

$$X(p) = \frac{1}{p} C_p^j.$$

Table 1

	1	3	5	7	9	11	13	15	17
2	1	1	1	1	1	1	1	1	1
4	1	2	3	4	5	6	7	8	9
6	1	4	7	12	19	26	35	47	57
8	1	5	14	30	55	91	140	204	285
10	1	7	26	66	143	273	476	776	1197
12	1	10	42	132	335	728	1428	2586	4389

Table 2

n_3	n_2	Lyndon words	Rhythmic patterns	Ethnic group
2	1	*ab*	332	Zande
	3	*abbb*	32322	Aka, Gbaya, Nzakara
	5	*abbbbb*	3223222	Gbaya, Ngbaka
	7	*abbbbbbb*	322232222	
	9	*abbbbbbbbb*	32222322222	Aka
4	1	*aab*	33332	
	3	*aabbb*	3323322	
		ababb	3323232	
	5	*aabbbbb*	332233222	
		ababbbb	332232322	
		abbabbb	323232322	
6	1	*aaab*	3333332	
	3	*aaabbb*	333233322	
		aababb	333233232	Aka (*mokongo*)
		aabbab	333232332	
		ababab	332332332	not primitive

Proof. The computation of a solution consists in placing j symbols equal to a in a word of length p over the alphabet $\{a, b\}$, and removing solutions that are cyclic shifts of one another. If n_3 is a power of 2, then $f(w)$ has exactly $|f(w)| = p$ cyclic shifts. Indeed, assume it has not, then $f(w)$ is a power of a shorter word, and since n_3 is a power of 2, $|f(w)|_b$ is even, thus one can write $f(w) = \gamma\gamma$. Then by Proposition 5, one has $\gamma\gamma(\varepsilon, \varepsilon) = \gamma(u, v) = (u, v).(u, v) = (uu, vv)$. It follows that $w = uuvv$ does not satisfy the rhythmic oddity property, since $h(uv) = h(vu)$. This implies that $f(w)$ has exactly $|f(w)| = p$ different cyclic shifts. □

Corollary 10. *If $n_3 = 8$, the number of solutions is the sum of the first squares.*

Proof. One has $j = 4$, thus

$$X(p) = \frac{(p-3)(p-2)(p-1)}{4!}.$$

Since n_2 is odd, one can write $n_2 = 2k - 1$. Then $p = n_2 + j = 2k - 1 + 4 = 2k + 3$. It follows

$$X(p) = \frac{k(k+1)2k+1}{6}$$

which proves that $X(p)$ is the sum of the k first squares. □

5. Results

The computation gives Table 2. Patterns actually used in Central African repertoires are indicated in the last column. There are reasons why those corresponding to the case $n_3 = 2$, $n_2 = 7$, and to the case $n_3 = 4$ are not used (see [5] for more details).

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